COMPUTATIONAL ISSUES IN THE STATISTICAL DESIGN AND ANALYSIS OF EXPERIMENTAL GAMES

Mahmoud A. El-Gamal, Richard D. McKelvey, and Thomas R. Palfrey

DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA 91125

Summary

One goal of experimental economics is to provide data to identify models that best describe the behavior of experimental subjects and, more generally, human economic behavior. We discuss here what we think are the three main steps required to make experimental investigations of economic games as statistically informative as possible: finding the solution of the experimental game under the postulated equilibrium or other economic models, selecting from a potential class of experimental designs the optimal one for discriminating between those models, and choosing an optimal stopping rule that indicates when to stop sampling data and accept one model as the best explanation of the data. Each step can be computationally intensive. We offer an algorithmic presentation of the necessary computations in each of the three steps and illustrate these procedures by examples from our research on learning models in experimental games with incomplete information. These three steps of experimental design and analysis are not limited to experimental games, but the computational burden of implementing these algorithms in other experimental environments—for example, market experiments—requires further considerations with which we have not dealt.

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Introduction

We shall take as primitives a class of theoretical hypotheses regarding the behavior of experimental subjects and a class of economic environments in which those hypotheses are to be tested. Our primary goal is to optimally design experimental studies to discriminate between the primitive collection of hypotheses and analyze the data produced by those experiments. There are a variety of reasons for optimizing the experimental design. The most obvious is that experiments are costly. Moreover, some experimental designs may discriminate between models so poorly as to render them useless, even if the budget constraints on experimental funds were not binding. We recognize three distinct steps for proceeding from the primitive class of possible experiments and class of rival theoretical models for predicting choice behavior in each of the possible experimental environments, to selecting an optimal experimental design.

The first step is to derive the statistical predictions of each of the rival hypotheses in each of the potential experimental designs. This involves computing for each model and experimental design a likelihood function that assigns to each possible data point which we can observe under our design the probability that that data point was generated by this particular model. This is typically not a simple exercise. Since simple economic models with perfectly rational agents often make predictions that are too sharp, the use of those models can lead to *the zero likelihood problem:* It is possible that we observe data that all of our models predict could never happen (e.g., subjects choosing strictly dominated actions). Ad hoc procedures such as discarding the zero-likelihood data points defy the likelihood principle and are incompatible with the application of the statistical approach to design optimization and data analysis that it requires. We therefore must adapt all of the models so that they assign positive probability (likelihood) to all possible data points. In our recent work (e.g., Boylan and El-Gamal, 1993; McKelvey and Palfrey, 1992; El-Gamal, McKelvey, and Palfrey, 1993a; El-Gamal and Palfrey, 1993), we have avoided the zero-likelihood problem by introducing an error structure which posits that subjects occasionally make errors. This seemingly

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minor change in the models typically creates a major computational challenge. In the absence of errors of this type, the models can usually be solved in closed form if the environments are simple enough. However, with error, in even the simplest environments, calculating the predictions of the competing models turns out to be a more difficult problem, which requires us to search for numerical solutions. We present below a general algorithm for numerically finding the predictions of a given model. We illustrate our algorithm to calculate a Nash equilibrium to the centipede game with a particular error structure introduced in El-Gamal, McKelvey, and Palfrey (1993b) and discuss the numerical techniques we used to find that equilibrium.

The next step that is required once we obtain the likelihood functions under each model, for each potential experimental design, is to find the optimal design—within the class under consideration—for discriminating between those models. This optimal design selection problem involves considering all possible data sets that can be observed, calculating likelihood ratios for each of the models under each of those data sets, and calculating the Kullback-Liebler information numbers for each design. We can then select the design that is most informative. The error structure of these models requires the introduction of several nuisance parameters under each model; hence the calculation of those information numbers has to be performed under given priors on those nuisance parameters. This can also be a rather challenging numerical problem, which we discuss below. We also provide a general algorithm for finding the most informative experimental design given our priors on the models and their nuisance parameters. We illustrate that algorithm with an application to the game of Vertigo introduced in El-Gamal and Palfrey (1993). We find the optimal design under uninformative priors on the nuisance parameters, then reoptimize the design after one experimental session, using the updated beliefs on the nuisance parameters.

The last set of calculations that we need to make deals with the analysis of the data as they are being collected. As we collect data from our chosen design, we update our beliefs about the various models and their nuisance parameters. Given a loss function for the rel-

ative cost of running more experiments versus possibly choosing the wrong model, we can arrive at an approximately optimal sequential probability ratio test (SPRT) that tells us whether we should stop sampling and accept one of the rival models. We provide below a general algorithm for this optimal sequential sampling and design problem. We illustrate the optimal sequential sampling by an application in El-Gamal, McKelvey, and Palfrey (1993a).

Solving for Fixed Points

Among the most common tools for finding solutions to economic problems are the fixed point theorems. For instance, the solution of a stochastic dynamic program (S, X, Q, r, β) with state space S, compact action space X, weakly continuous state transition Q, continuous and bounded reward function r, and a discount factor $\beta \in (0,1)$, is obtained by solving for the value function:

$$V(s) = \max_{x \in X} \left\{ r(x,s) + \beta \int_S V(s')Q(ds'|x,s) \right\}.$$

The existence of such a value function uses the fact that the operator T, defined by

$$TV(s) = \max_{x \in X} \left\{ r(x,s) + \beta \int_S V(s')Q(ds'|x,s) \right\},$$

satisfies monotonicity and discounting, and hence is a contraction mapping (Blackwell, 1965). Since T is a contraction mapping, if we start with any initial guess V_0, and iterate $V_{n+1} = TV_n$, we are guaranteed that $V_n \to V$, the unique fixed point of the T. Implementing this iterative algorithm for finding the fixed point by evaluating the functions $V_n(s)$ at a finite number of grid points is a common technique for solving problems of this class.

In game theory with incomplete information, the solution concept of a symmetric (Bayes) Nash equilibrium is also a fixed point concept. For a game of incomplete information with n players, action space A (the same for all players), and payoffs $\pi_i(a_1, \ldots, a_n)$ to player i, a strategy is simply a function σ: $\Lambda \to \mathcal{P}(A)$ (where Λ is the space of beliefs on the relevant parameters of the game, and $\mathcal{P}(A)$ is the space of probability measures on A). In the extensive form, the strategy will be a mapping for each stage s of the game σ_s: $\Lambda \times H_s \to \mathcal{P}(A_s)$, where H_s is the history up to stage s, and A_s is the action space for stage s. As the two representations can equally serve our purpose, we shall concentrate on the normal form of the game. Given $n - 1$ players are using a vector of strategies $\vec{\sigma}'$, and assuming that the beliefs of the n players $(\lambda_1, \ldots, \lambda_n)$ are drawn from a common knowledge distribution μ, the best response of the nth player is defined by

$$\sigma_{BR}(\lambda, \vec{\sigma}') \in \arg\max_{\sigma \in \Sigma} \sum_{a_1, \ldots, a_n} \int_{\lambda_1, \ldots, \lambda_{n-1}} \int_S \prod_{i=1}^{n-1} \sigma_i'(\lambda_i)(a_i)\ \sigma(\lambda_n)(a_n)\ \pi_n(a_1, \ldots, a_n; s)\ \lambda(ds)\ \mu(d\lambda_1, \ldots, d\lambda_{n-1}).$$

A symmetric Bayes Nash equilibrium σ is a strategy that is the best response to itself. In other words, if we let the $n - 1$ players use the same strategy σ, then $\sigma \in \sigma_{BR}(\lambda, \sigma, \ldots, \sigma)$. Finding a Bayes Nash equilibrium is, therefore, a fixed point problem. Unfortunately, unlike the case of discounted dynamic programming, the best response operator usually does not have the nice property of being a contraction mapping. We therefore need an algorithm to find the fixed point that does not impose such structure on the mapping.

The solution lies in recognizing that our fixed point problem can be converted into a minimization problem. Let us consider an arbitrary fixed point problem: For a given operator T: $\mathcal{F} \to \mathcal{F}$, find f such that $Tf = f$. Note that by constructing the functional $Q(f) = \min_{f \in \mathcal{F}} \|Tf - f\|$, the function $Q(f) \geq 0$, and $Q(f) = 0$ if and only if f is a fixed point of T. In the two applications that we have mentioned, the function f could be viewed as a vector in $\mathcal{R}^d$. In the dynamic programming example, this interpretation corresponds to the value function being evaluated at a finite number (d) of grid points. In our games with incomplete information example, it corresponds to A being finite (with d actions available to each of the players); hence a strategy at a given λ is simply a vector of probabilities $(p_1, \ldots, p_d) \in [0, 1]^d$. In either case, the problem is one of finding a fixed point of some operator T_d on $\mathcal{R}^d$ and can be converted into a problem of minimizing the multidimensional function

$$Q_d(x_1, \ldots, x_d) = \min_{y_1,\ldots,y_d} \|T_d(y_1, \ldots, y_d) - (y_1, \ldots, y_d)\|.$$

This minimization problem can be numerically solved using any one of many standard multivariate minimization routines.

ALGORITHM FOR FINDING EQUILIBRIUM

In the particular example of finding a symmetric Bayes Nash equilibrium for a game with incomplete information, we can use the extra structure provided by the game to simplify the minimization problem even further. For instance, assume that there is a given belief λ which is a common knowledge belief (i.e., all players have the same belief, they know that everyone else knows that they all have the same belief, they know that everyone knows that . . .). Then, a strategy for that fixed λ is simply a vector $(p_1, \ldots, p_d) \in [0, 1]^d$. We can now construct the function:

$$Q(p,\lambda) = \sum_{p' \in \{0,1\}^d} (\max[v(p',p,\lambda) - v(p,p,\lambda),0])^2,$$

where

$$v(p',p,\lambda) = \sum_{a_1,\ldots,a_n} \int_S \prod_{i=1}^{n-1} p_i(a_i)\, p'(a_n)\, \pi_n(a_1, \ldots, a_n; s)\lambda(ds)$$

is the value to the nth player of using strategy p' when all other players are using strategy p. It is straightforward (see McKelvey, 1990) to show that the function Q is nonnegative and differentiable, with $Q(p) = 0$ if and only if p is a symmetric Bayes Nash equilibrium. For each belief λ, action space $A = a^1, \ldots, a^d$, and payoffs $\pi(.)$, this suggests the following simple algorithm for finding the symmetric Bayes Nash equilibria:

```
Choose initial guess p, tolerance level t for Q, etc.
Call gradient minimization algorithm for Q with parameters
p, t, etc.
. . .
. . .
Subroutine for evaluation of Q(p)

evaluate v (p,p)
funct = 0
Loop over all pure strategies p'
    evaluate v(p',p)
    diff = v(p',p) - v(p,p)
    if (diff < 0) diff = 0
    funct = funct + diff^2
end loop
return funct
```

ILLUSTRATION: CENTIPEDE GAME

We illustrate the algorithm discussed in the previous section with the solution of a centipede game with irrational agents from El-Gamal, McKelvey, and Palfrey (1993b). Our game has four players: two of them are labeled red players, and two are labeled blue. Each player plays (in sequence) two stages of the game in Figure 1, each stage being played with each of the two players of the other color. In each stage game, the red player begins with an opportunity to take \$4 and leave the blue player \$1, or pass. If the red player passes, then the blue player can take \$8 and leave the red player \$2, or he can pass. If the blue player passes, then the red player has another turn, and can take \$16 and leave the blue player \$4, or can pass again. If the red player passes at this last move of the stage game, then the blue player gets \$32, and the red player gets \$8, and this stage game is over.

If all players were perfectly rational, then all four stage games would end in the red players taking on the first round. If perfect rationality is not assumed, the problem becomes much more complicated (e.g., see Binmore, 1989; Reny, 1993; Rosenthal, 1982; Aumann, 1988; Kreps et al., 1982). We propose a particular type of irrationality: namely, that an irrational player always flips a fair coin to choose whether he should pass or take at each of his decision nodes. We assume that there is a true proportion q of irrationals in the population, each agent i has a belief $U[0,\delta_i]$ on q, and believes that everyone else's belief on q is identical to his. To symmetrize the game, and hence be able to look for a symmetric Bayes Nash equilibrium, we assume that nature randomly chooses the players' colors with equal probability, and irrationality independently with probability q. Thus, a rational player has to decide on a strategy if red and a strategy if blue. A strategy for red consists of one probability (mixed strategy) for tak-

ing in the first move of the stage game, and four probabilities for taking in the first move of the second stage game, one for each of the four possible histories in the first stage game: red takes (T), red passes then blue takes (PT), red passes then blue passes then red takes (PPT), or red passes then blue passes then red passes (PPP). A rational red player will always take in the last move of either stage game, so those strategies are suppressed. Similarly, the strategy for blue consists of one probability for taking in the first stage game and four probabilities for taking in the second stage game, depending on whether the first stage game ended in T, PT, PPT, or PPP. The full symmetric Bayes Nash equilibrium of this game as modeled, therefore, is 10 probabilities (5 for red, 5 for blue) for each value of the belief $\delta \in [0, 1]$, and the equilibrium fixed point can be obtained by using the algorithm in the previous section to minimize:

$$Q(p;\delta) = \sum_{p' \in \{0,1\}^{10}} (\max[v(p',p;\delta) - v(p,p;\delta),0])^2.$$

We refer to the resulting equilibrium as the Sequential-Nash (SN) equilibrium. The alternative equilibrium model we used is one where the agents play each of the two stage games ignoring the fact that they play the other. In that second model, labeled the Myopic Nash (MN), players do not update about q between the two games; the minimization problem for finding the equilibrium of the MN model is two-dimensional (probability of red passing and probability of blue passing).

For the calculation of the equilibria depicted in Figures 2 through 7, we implemented the above algorithm at each of 30 grid points on $\delta \in [0, 1]$ and used the Powell's method routine in Press et al. (1988) to bring $Q(p; \delta)$ to less than 10^{-20} at each of those grid points. The rest of the program was written in C and compiled and vectorized on a CRAY X-MP/18. We needed approximately 40 CPU hours to achieve the final solutions reported here. In our experience with this problem, the function $Q(.; \delta)$ was not very well behaved, in the sense of having many narrow valleys far away from the zero (minimum) of the function. This suggests that nongreedy algorithms, such as simulated annealing, will be

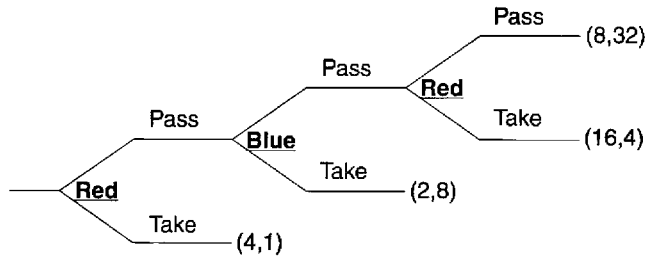

Fig. 1 One stage of the three-move centipede game

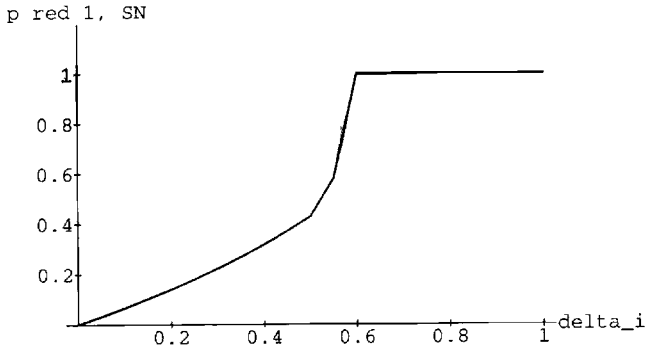

Fig. 2 Prob. red passes first game, SN

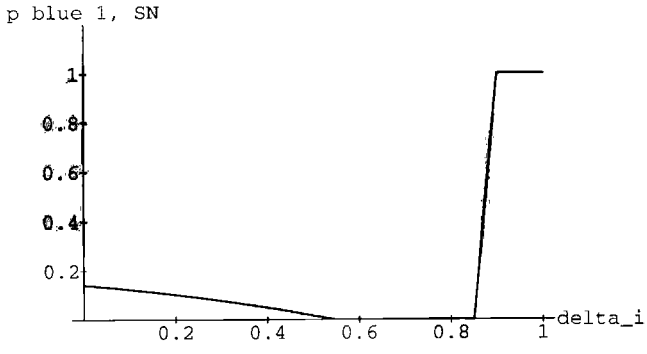

Fig. 3 Prob. blue passes first game, SN

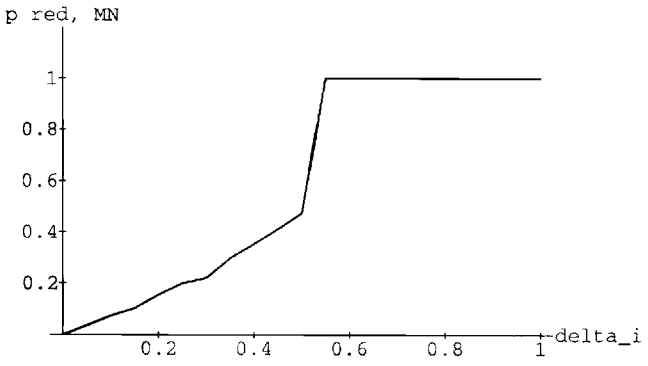

Fig. 4 Prob. red passes each game, MN

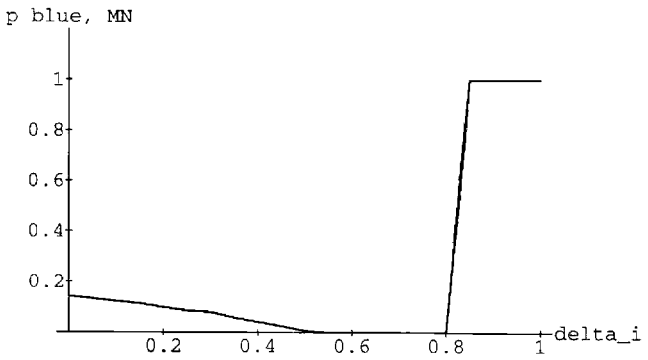

Fig. 5 Prob. blue passes each game, MN

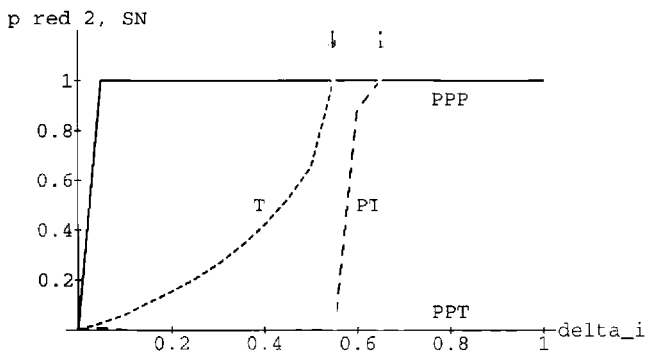

Fig. 6 Prob. red passes second game, SN

poorly suited for locating the fixed point strategies (equilibria) for similar problems.

Optimal Experimental Design

In the previous section, we discussed the general numerical methodology of finding equilibria to experimental games under models that allow all observable data samples to occur with positive probability. Now, given the class of models in question, the equilibria under these models give us likelihood functions for observable data from a variety of experimental designs. In the experimental design stage, we can invoke statistical notions of optimal design to discriminate between the given class of models. To make our discussion concrete, let us assume that we have a class of experiments parameterized by some parameter vector α. Typically, α will correspond to payoff structures, probability distributions on payoff relevant state spaces, etc. Then, the optimal experimental design problem is the problem of finding the optimal value of α to use in our experiments.

Let X be the space of all possible data sets under all of our designs. Denote a typical data set by x. Let the likelihoods of a given data set $x \in X$ under design α for each of our n competing models be $l_1(x;\alpha), \ldots, l_n(x;\alpha)$. Given a collection of priors on models $1, \ldots, n$, say $p_1, \ldots, p_n$, we can define for each model the Kullback-Liebler information number measuring how informative a given design is expected to be if that model were correct. For example, the information number for model 1 under design α is simply

$$I(1;\alpha) = \sum_{x \in X} l_1(x;\alpha) \log \left(\frac{(1 - p_1) l_1(x;\alpha)}{\sum_{i=2}^{n} p_i l_i(x;\alpha)} \right).$$

The design that maximizes our expected separation between model 1 and the other $n - 1$ models, if model 1 were indeed the correct model, is $\alpha_1^* = \arg\max_{\alpha \in A} I(1;\alpha)$. If we want to maximize the overall informativeness of our design, we weight our information numbers by our prior on each of the models and choose $\alpha^* = \max_{\alpha \in A} \Sigma_{i=1}^{n} p_i I(i;\alpha)$.

ALGORITHM FOR OPTIMIZING THE DESIGN

Starting with a family of likelihood functions $l_1, \ldots, l_n$ corresponding to our n models, priors $p_1, \ldots, p_n$ on the n models, and a family A of designs, we need to calculate the information number for each of the models

$$I(i;\alpha) = \sum_{x \in X} l_i(x;\alpha) \log \left(\frac{(1 - p_i) l_i(x;\alpha)}{\sum_{j \neq i} p_j l_j(x;\alpha)} \right),$$

finding the optimal design $\alpha_i^* = \max_{\alpha \in A} I(i;\alpha)$ for each of our models, and the overall optimal design α^* defined in the previous subsection. The following algorithm accomplishes this.

```
Benchmark_1 = . . . = Benchmark_n = Benchmark = 0
Loop over all designs a in A
    Info_1 = . . = Info_n = 0
    Loop over all possible data sets x in X
        Calculate likelihoods l_1(x;a), . . . , l_n(x;a)
        Loop over i = 1, . . . , n
            denom = 0
            Loop over j ! = i
                denom = denom + p_j * l_j (x;a)
            End Loop over j
            Info_i = Info_i + l_i(x;a) * log[(1 - p_i) *
            l_i(x;a)/denom]
        End Loop over i
    End Loop over x
    info = 0
    Loop over i = 1, . . . , n
        Info = Info + p_i * Info_i
        if (Info_i ≧ Benchmark_i) Benchmark_i = Info_i;
        design_i = a
    End Loop over i
    if (Info ≧ Benchmark) Benchmark = Info; design = a
End Loop over a
```

ILLUSTRATION: VERTIGO

Obviously, for complicated games, the procedure outlined above would be computationally infeasible. In order to implement our optimal experimental design algorithm of the previous subsection, it must be relatively easy to calculate the likelihood functions $l_n(x;\alpha)$ for all designs parameterized by $\alpha \in A$. We therefore chose a class of games that are parameterized by a small number of parameters and whose equilibria are easy to find for each of the rival models. For that purpose, we introduced the game of Vertigo in El-Gamal and Palfrey (1993). Vertigo is a game of one-sided incomplete information. The stage game is defined by the following pair of possible games, labeled Game I and Game II:

Game I, drawn with probability π

	L	R
U	$(0,a_2)$	$(a_2,0)$
D	$(a_1,0)$	$(0,a_1)$

Game II, drawn with probability $1 - \pi$

	L	R
U	$(a_2,0)$	$(0,a_1)$
D	$(0,a_2)$	$(a_1,0)$

The entries in each box are the payoffs to row player and column player, respectively, corresponding to a particular strategy profile. The players know the values of π and that Game I is drawn with probability π. The row players observe whether Game I or Game II was drawn, but the column players do not. Each of the component games has a unique mixed strategy equilibrium, and the only relevant payoff statistic for all players is $b = a_1/(a_1 + a_2)$. The general class of designs was fixed, with π and b left as design parameters over which we can optimize. In the notation of the previous subsection, $\alpha = (\pi, b)$. The game is played in five stages and 10 rounds. In the beginning of each round, a game (Game I or Game II) is drawn with the announced probability π and is used for the five stages of that round. All the row players are informed of the draw for that round. In each of the stages, the row and column players move simultaneously, and they are informed of their opponent's move, but the column players are not informed of their payoffs (and hence cannot directly infer which game was drawn). For each stage, each column player is matched with a row player he has not played before, and the row player he is currently matched with is informed of the history of moves by the other row players who have played his current match. This allows the row player to infer the current belief (about which game was drawn) of the column player with whom he is matched. After all five stage games have been played, the payoff table is revealed to the column players, and they calculate their payoffs. The

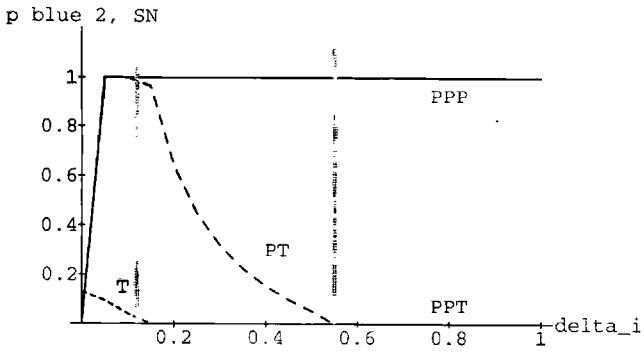

Fig. 7 Prob. blue passes second game, SN

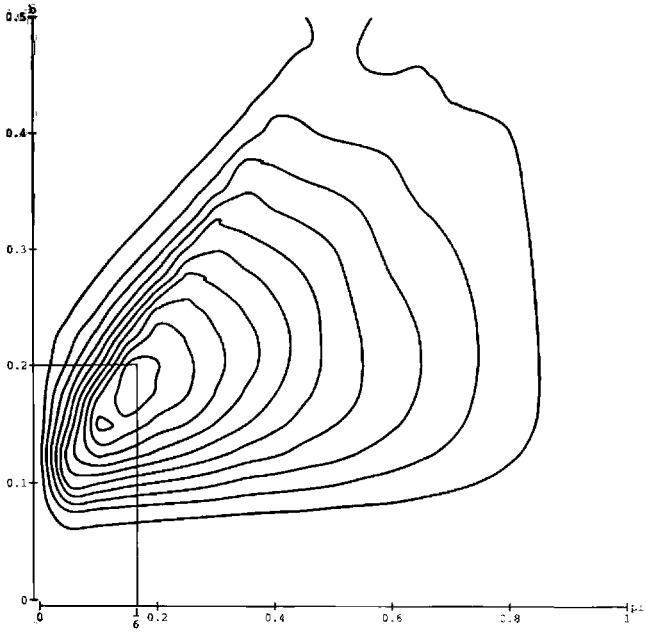

Fig. 8 Optimal design with priors $\epsilon_0 \sim U[0,1]$, $\kappa \sim U[0,0.3]$

entire procedure is then repeated for later rounds, with a new game being drawn.

In El-Gamal and Palfrey (1993), we had six models of behavior corresponding to varying degrees of sophistication in updating and strategic behavior of the subjects. We shall concentrate here on two of those models to illustrate the issues of optimal experimental design. In both these models, we assume that there is a probability ϵ_t with which each of the players in each of the stage games in round t can become confused and choose an action by flipping a fair coin. We assume that there is learning by doing, which causes the probability of confusion to decline over time, so we model $\epsilon_t = \epsilon_0 e^{-\kappa t}$. The values of ϵ_0 and κ are assumed to be common knowledge for the subjects, but unknown to us.

The two models we consider here differ in that one allows the subjects to use Bayesian updating between rounds to learn about the true game that is being played, and the other does not. Of the six models in El-Gamal and Palfrey (1993), these models are labeled UF for unsophisticated with fast updating and UN for unsophisticated with no updating. We impose priors on the nuisance parameters (ϵ_0, κ) by first deciding on the supports being $\epsilon_0 \in [0, 1]$ and $\kappa \in [0, 0.3]$ ($\kappa = 0.3$ corresponds to $\epsilon_{10} \approx \epsilon_0/20$), and then making our priors uniform over those supports. Using those priors and the equilibrium behavior under our two models, we can construct the likelihood functions for all possible data sets and compute the information numbers as discussed earlier in this section. The optimal design for distinguishing between our two models was found to be $\pi = 1/6$ and $b = 0.2$. The contours of the information surface as a function of (π, b) between the two given models are shown in Figure 8.

After running an experimental session with 16 subjects, the posterior odds ratio between those two models was still very close to 1, but our posteriors on ϵ_0 and κ had changed considerably, concentrating most of the mass near $\epsilon_0 = 0.7$ and $\kappa = 0.1$. When we recomputed the information surface with those new parameters, the optimal design turned out to be $\pi = 0.2$ and $b = 0.4$. The contours of the information surface with the posteriors on ϵ_0 and κ are shown in Figure 9. We used that second design in an experimental session with 10 sub-

jects, and the result was a posterior odds between the two models of the order 10^5 in favor of the model with no updating. In fact, we found the unsophisticated model with no updating to be the best of *all* six models we analyzed in El-Gamal and Palfrey (1993), significantly outperforming all the sophisticated and unsophisticated models in explaining the behavior of our experimental subjects. Note that despite the fact that the first design did not help us distinguish between the two models, its by-product updating of our priors on the nuisance parameters allowed us to recalculate the information numbers, and the reoptimized design gave us the very sharp results that we were seeking. The most computationally intensive part of our analysis was the calculation of the information numbers over a grid of 50 points on π and 50 points on b. Each of the design surfaces whose contours are shown in Figures 8 and 9 required approximately 2 CPU hours to complete on a CRAY Y-MP/2E/116.

Optimal Sequential Sampling

One problem faced when collecting experimental data is how to decide when to stop sampling. One simple decision rule we can use is to allocate a fixed budget for a particular series of experiments and stop sampling when the budget is exhausted. This decision rule, however, is not based on any optimality principles. For example, it is conceivable that early data could be so decisive in discriminating between the rival models that the rest of the budget can be saved for other projects. It is also conceivable that after collecting some data, our beliefs about the nuisance parameters of the various models change in such a way that we know that a particular experimental design will not be useful in discriminating between the models of interest. We have already seen the latter effect in the illustration in the previous section, where the *ex ante* optimal design turned out to be quite uninformative once our beliefs on the nuisance parameters were updated. In this section, we wish to disentangle the issues of optimal design and redesign from the issue of finding an optimal stopping rule. Therefore, in the remainder of this section, we assume that we have a given design with which we sample until we decide to stop and accept one of the rival models. For simplicity, we shall, for the remainder of the section, consider the case with only two rival models.

The stopping rule we chose belongs to the family of Wald's sequential probability ratio tests (SPRTs). If we induce a 0-K loss function (the loss of selecting the correct model is 0, the cost of selecting the wrong model is K), the SPRT has the remarkable property of minimizing the expected sample size (and hence the expected cost of the set of experiments) in the class of all tests with the same type I and type II error probabilities (e.g., see Chernoff, 1972, chapters 11,12, and Berger, 1985, chapter 7). SPRTs take the form: continue sampling until the likelihood ratio between the two models crosses one of two boundaries. If the upper boundary is crossed, accept the model whose likelihood appears in the numerator of the likelihood ratio, and if the lower boundary is the one crossed, accept the other model. A number of approximations to compute the optimal stopping boundaries have been proposed. The most popular approximations involve using Wald's approximation of probabilities of type I and type II error and the expected stopping time (Chernoff, 1972, pp. 59–66; Berger, 1985, pp. 485–499). Berger (1985, p. 500) suggests a further approximation based on the cost per experiment c being much smaller than the loss of selecting the wrong model K. The resulting rule sets the boundaries $A = -c\pi/(I(2)K(1 - \pi))$, and $B = \pi KI(1)/(c(1 - \pi))$, where π is the prior on model 1 being correct, $I(i)$ is the information number for model i calculated in the same manner as the algorithm in the previous section prescribes, and we stop and accept model 1 if the likelihood ratio of model 1 to model 2 is greater than B, stop and accept model 2 if the likelihood ratio is less than A, and continue sampling otherwise.

ALGORITHM FOR OPTIMAL SEQUENTIAL SAMPLING

In this section, we consider two rival models and a given experimental design, and the only issue is whether we should run another experiment or not. The following algorithm outlines the main steps of calculating the stopping boundaries of the SPRT discussed above. It is

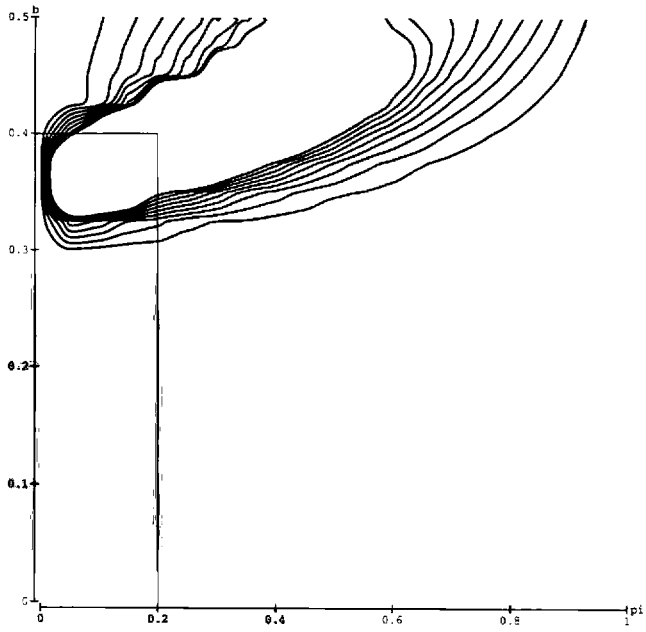

Fig. 9 Optimal design with posteriors on $\epsilon_0 \approx 0.7$, $\kappa \approx 0.1$

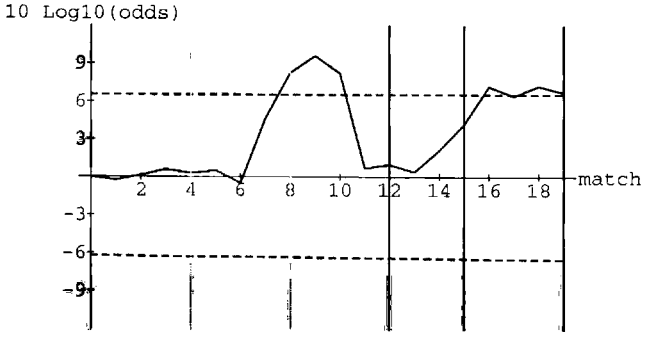

Fig. 10 Time series of 10 $\log_{10}$ of odds ratio (solid line) and stopping boundaries (dashed lines)

easy to see how we can implement more sophisticated versions of this algorithm which take into consideration the possibility of sequentially optimizing and reoptimizing the design.

```
Choose cost per experiment c
Choose loss for wrong decision K ⩾ c
Choose prior odds = pi in favor of model 1
Follow steps in Algorithm for optimal design to compute
l(1) and l(2)
A = −c * pi/(l(2) * K(1 − pi))
B = pi * K * l(1)/(c *(1 − pi))
Flag = 0
Sample until Flag = 1
    Calculate l_1(x) and l_2(x)
    odds = odds * l_1(x)/l_2(x)
    if (odds ≧ B) Flag = 1, accept model 1
    if (odds ≦ A) Flag = 1, accept model 2
if (A < odds < B) continue
```

ILLUSTRATION: CENTIPEDE GAME

In El-Gamal, McKelvey, and Palfrey (1993a), we implemented the SPRT described above for analyzing a slightly more complicated model of the centipede game presented above. In that model, we assumed that there was a fixed proportion q of altruists in the population. Altruists always pass when given the chance. Each agent i was modeled as having a belief about q which is $U[0, \delta_i]$ and was assumed to believe that all other agents had the same δ_i. The true distribution of the δ_i's was assumed to be $U[0, \gamma]$, and γ is unknown to us. We also assumed that with probability ϵ, each of the players at each of the nodes could get confused and decide whether to pass or take by flipping a fair coin. The probability ϵ was assumed to be common knowledge for the players, but unknown to us. For each pair of parameters ϵ and δ, we can solve for the symmetric Bayes Nash equilibrium of the game following the algorithm for finding equilibrium given above. We called the model in which all the players use the strategies defining that symmetric Bayes Nash equilibrium the Sequential Nash model. The rival model we posited is the one of McKelvey and Palfrey (1992), in which the symmetric Bayes Nash equilibrium is found for the one-stage game and is assumed to be the same for all stages (i.e., that players do not update their beliefs about q between games and do not take into

consideration the possibility of such updating in constructing their strategies), and we called that the Nonsequential Nash model.

For the SPRT between the Sequential and Nonsequential Nash models described above, we chose the values $K = 1$ and $c = 0.01$, and started with the unbiased prior odds $\pi = \frac{1}{2}$ in favor of the SN model (an initial odds ratio of 1). Using those parameters in the above-described algorithm yields stopping boundaries $A = 0.238$ and $B = 4.5$. After sampling 19 matches (each consisting of four players as described above), we stopped and chose the Sequential Nash model. The dynamics of our posterior odds ratio is shown in Figure 10. To make the plot more symmetric, we plot 10 times logarithm of base 10 of the posterior odds ratio and stopping boundaries. The vertical lines correspond to experimental sessions, and the gray lines indicate that the first three sessions were inherited from an earlier design. We had started with the model of rationality discussed above but found that the given design, together with those models, was uninformative. We therefore switched to the model of rationality described in this section, reanalyzed the old data, and continued sampling until we accepted one of the two models. For more details on the decision to change our models of irrationality, see El-Gamal, McKelvey, and Palfrey (1993b). The most computationally intensive part of this study was finding the equilibria for a grid of 30 points on δ and 20 points on ϵ, under the new model of irrationality, which took approximately 20 CPU hours on a CRAY Y-MP/2E-116. To find the equilibria under the new model of irrationality, we first initialized the search for the equilibrium with $\epsilon = 0.05$ at the equilibrium under the old model of rationality. For each higher ϵ, we initialized the search at the equilibrium for the next lowest ϵ on the grid. The calculation of the information numbers for this model was not very computationally intensive because our design allowed for only 256 possible data sets.

Concluding Remarks

We have defined what we view as the three most important computational aspects in designing and analyzing experimental studies of games: finding the likelihood functions under each of a class of rival models, finding the optimal experimental design for discriminating between those models, and finding the optimal rule for stopping and accepting one of the models. We discussed each of the three issues in a general context that makes our discussion, and the algorithms we propose, applicable to a very wide class of studies (not necessarily limited to experimental games or to experimental studies in general). We then illustrated the feasibility and usefulness of implementing those algorithms with applications in our research on learning in experimental games. We hope that as the computational restrictions continue to be relaxed, the issues of optimal statistical design and analysis will be more seriously considered in future experimental economic research.

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BIOGRAPHIES

Mahmoud A. El-Gamal is an assistant professor of economics at the California Institute of Technology. Before arriving at Caltech, he was an assistant professor of economics at the University of Rochester. He received his M.S. in statistics from Stanford University in 1985 and his Ph.D in economics from Northwestern University in 1988.

Richard D. McKelvey is a professor of political science at the California Institute of Technology. Before arriving at Caltech, he held positions as assistant professor at the University of Rochester and as assistant and associate professor at Carnegie-Mellon University. In 1978–79, he held the position of Sherman Fairchild Distinguished Scholar at Caltech. He has been recently awarded

membership in the National Academy of Sciences. His recent research topics include theoretical and experimental investigations of games of incomplete information, computation of game-theoretic equilibria, and political business cycle growth models.

Thomas R. Palfrey is a professor of economics and political science at the California Institute of Technology. Before arriving at Caltech, he was professor of political economy at the Graduate School of Industrial Administration at Carnegie-Mellon University. He has visited as a fellow at the Center for Advanced Study in Behavioral Sciences, and has held visiting professorships of economics at University of Toulouse and Texas A&M University. His current research focuses on theoretical and experimental studies of games, with applications to electoral competition and mechanism design.

SUBJECT AREA EDITOR

John Rust

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